HW6 (extra) due Friday, May 3

Turn in 12.6#8eg

Practice 5.3#4, 5.3#8, 12.3#3, ~~12.3~~#bd, 12.6#7a, 12.6#9a

(Skip 12.5 in the book...)

should be

12.2#5 bd

12.2.5. Solve using the method of characteristics (if necessary, see Section 12.6):

(a) $\dfrac{\partial w}{\partial t} + c\dfrac{\partial w}{\partial x} = e^{2x}$ with $w(x,0) = f(x)$

*__(b)__ $\dfrac{\partial w}{\partial t} + x\dfrac{\partial w}{\partial x} = 1$ with $w(x,0) = f(x)$

(c) $\dfrac{\partial w}{\partial t} + t\dfrac{\partial w}{\partial x} = 1$ with $w(x,0) = f(x)$

*__(d)__ $\dfrac{\partial w}{\partial t} + 3t\dfrac{\partial w}{\partial x} = w$ with $w(x,0) = f(x)$

Example:

linear PDE... don't expect shocks...

*__(b)__ $\dfrac{\partial w}{\partial t} + x\dfrac{\partial w}{\partial x} = 1$ with $w(x,0) = f(x)$

$$\frac{d}{dt} w(x(t),t) = \frac{\partial w}{\partial t} + x'(t)\frac{\partial w}{\partial x}$$

ODEs:

$$x'(t) = x \quad \text{and} \quad \frac{dw(x(t),t)}{dt} = 1$$

$$\begin{cases} x(t) = x_0 e^t \\ x(0) = x_0 e^0 = x_0 \end{cases}$$

$$w(x(t),t) = t + \text{const}$$

$$w(x_0,0) = 0 + \text{const} = f(x_0)$$

$$w(x(t),t) = t + f(x_0)$$

along the characteristics the solution is

$$w(x_0 e^t, t) = t + f(x_0)$$

Now set $x = x_0 e^t$ $\quad x_0 = x e^{-t}$

$$\boxed{w(x,t) = t + f(xe^{-t})}$$ solution

Work this problem...

*(d) $\frac{\partial w}{\partial t} + 3t\frac{\partial w}{\partial x} = w$ with $w(x,0) = f(x)$

ODEs

$x'(t) = 3t$ $\qquad$ $\frac{d}{dt} w(x(t),t) = w$

$x(t) = \frac{3}{2}t^2 + x_0$ $\qquad$ $w(x(t),t) = w(x_0,0)\, e^t$

Thus. $\boxed{w(\underbrace{\tfrac{3}{2}t^2 + x_0}_{x = \frac{3}{2}t^2 + x_0}, t) = f(x_0)\, e^t}$ implicit soln along characteristics

$x_0 = x - \frac{3}{2}t^2$

Solution $\boxed{w(x,t) = f(x - \tfrac{3}{2}t^2)\, e^t}$

2. Recall the one-dimensional heat equation with constant thermal properties given by

$$\cancel{c\rho \frac{\partial u}{\partial t}} = K_0 \frac{\partial^2 u}{\partial x^2} + Q \quad \text{for} \quad t \geq 0 \quad \text{and} \quad x \in [0, L].$$

Here c is the heat capacity, ρ the density, K_0 the conductivity, Q the rate of production of heat energy and u the temperature. Suppose $L = 2$ and $Q/K_0 = 1$. If the initial and boundary conditions satisfy

$$u(x, 0) = \cos(\pi x) \quad \text{for} \quad x \in [0, 2]$$
$$u(0, t) = 3 \quad \text{and} \quad u(2, t) = 1 \quad \text{for} \quad t > 0,$$

find the equilibrium temperature of the rod obtained as $t \to \infty$.

ODE

$$K_0 \frac{\partial^2 u}{\partial x^2} = -Q \qquad \frac{d^2 u}{dx^2} = -\frac{Q}{K_0} = -1$$

$$\boxed{\frac{d^2 u}{dx^2} = -1}$$

$$\frac{du}{dx} = \int -1\, dx = -x + c$$

$$\boxed{u = \int(-x + c)\, dx = -\frac{x^2}{2} + cx + d}$$

Now solve for c and d using boundary

$$u(0) = 3 = d \qquad \text{so} \quad d = 3$$

$$u(2) = 1 = \frac{-4}{2} + c2 + 3 \qquad c = \frac{1}{2}(1 + 2 - 3) = 0$$

Answer: equilibrium state

$$u(x) = -\frac{x^2}{2} + 3$$

Insulating boundary condition

$$\frac{\partial u}{\partial x}(0,t)=0 \qquad \frac{\partial u}{\partial x}(2,t)=0$$

As before

$$\frac{d^2u}{dx^2} = -1$$

$$\frac{du}{dx} = \int -1\,dx = -x + c$$

$$u = \int (-x + c)\,dx = -\frac{x^2}{2} + cx + d$$

now use bndry to solve for c and d.

$$u'(x) = -x + c$$

$$u'(0) = -0 + c = 0 \quad \text{so} \quad c = 0$$

$$u'(2) = -2 \neq 0$$

oops no equilibrium solution.