**3.** Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad t \geq 0 \quad \text{and} \quad x \in [0, L]$$

subject to the homogeneous boundary conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \quad \text{and} \quad u(L, t) = 0.$$

Solve the initial value problem if the temperature is initially

solve by superposition

$$u(x, 0) = -2\cos\left(\frac{5\pi x}{2L}\right).$$

Separation of variables $u(x,t) = h(t)\,\varphi(x)$ substitute

$$h'(t)\,\varphi(x) = k\,h(t)\,\varphi''(x)$$

$$\frac{h'(t)}{k\,h(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda$$

The ODEs are

$$h'(t) = -k\lambda h(t) \qquad \text{and} \qquad \varphi''(x) = -\lambda\varphi(x)$$

$$\varphi'(0) = 0 \qquad \varphi(L) = 0$$

Solve this ODE first

Case $\lambda < 0$. Then $\varphi(x) = a\cosh\sqrt{|\lambda|}\,x + b\sinh\sqrt{|\lambda|}\,x$

$$\varphi'(x) = a\sqrt{|\lambda|}\sinh\sqrt{|\lambda|}\,x + b\sqrt{|\lambda|}\cosh\sqrt{|\lambda|}\,x$$

$$\varphi'(0) = b\sqrt{|\lambda|} = 0 \quad \text{so } b = 0$$

$$\varphi(L) = a\cosh\sqrt{|\lambda|}\,L = 0 \quad \text{so} \quad a = 0$$

No eigenfunctions.

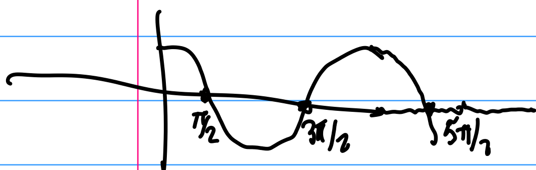
Case $\lambda = 0$. Then $\varphi''(x) = 0$

$$\varphi(x) = ax + b$$
$$\varphi'(x) = a$$
$$\varphi'(0) = a = 0 \quad \text{so} \quad a = 0$$
$$\varphi(L) = b = 0 \quad \text{so} \quad b = 0$$

No eigenfunction

Case $\lambda > 0$. Then $\varphi(x) = a\cos\sqrt{\lambda}\,x + b\sin\sqrt{\lambda}\,x$

$$\varphi'(x) = -a\sqrt{\lambda}\sin\sqrt{\lambda}\,x + b\sqrt{\lambda}\cos\sqrt{\lambda}\,x$$




$$\varphi'(0) = b\sqrt{\lambda} = 0 \quad \text{so} \quad b = 0$$
$$\varphi(L) = a\cos\sqrt{\lambda}\,L = 0$$

so $\sqrt{\lambda}\,L = (n + \frac{1}{2})\pi$ for $n = 0, 1, 2, \ldots$

$$\sqrt{\lambda} = \frac{(n+1/2)\pi}{L} \qquad \lambda = \frac{(n+1/2)^2\pi^2}{L^2}$$

Eigenfunctions $\varphi_n(x) = \cos\left(n+\frac{1}{2}\right)\frac{\pi x}{L}$

The other ODE

$$h'(t) = -k\lambda h(t)$$
$$h(t) = a e^{-k\lambda t}$$

so $h_n(t) = e^{-k(n+1/2)^2\pi^2 t/L^2}$

By superposition

$$u(x,t) = \sum_{n=0}^{\infty} a_n \varphi_n(x) h_n(t)$$
$$= \sum_{n=0}^{\infty} a_n \cos\frac{(n+1/2)\pi x}{L} e^{-k(n+1/2)^2\pi^2 t/L^2}$$

Use the superposition to solve for the initial condition

$$u(x,0) = \sum_{n=0}^{\infty} a_n \cos \frac{(n+1/2)\pi x}{L} = -2\cos\left(\frac{5\pi x}{2L}\right)$$

this is actually one of the eigenfunctions

$n=2$

so using orthogonality is easy in this case.

so $a_2 = -2$

Also...

$a_0 = 0$, $a_1 = 0$,

$a_n = 0$ for $n \geq 3$.

Therefore,

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos \frac{(n+1/2)\pi x}{L} e^{-k(n+1/2)^2\pi^2 t/L^2}$$

$$= a_2 \cos \frac{(2+1/2)\pi x}{L} e^{-k(2+1/2)^2\pi^2 t/L^2}$$

$$\boxed{u(x,t) = -2\cos \frac{5\pi x}{2L} e^{-k(5/2)^2\pi^2 t/L^2}}$$

typo fixed after class..

**6.** Consider the wave equation

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + \alpha u$$

where $\rho_0$, $T_0$ and $\alpha$ are constants subject to the homogenous boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0.$$

Solve the initial value problem if $\alpha < 0$ and

$$u(x,0) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = f(x).$$

Separation of variables. $u(x,t) = \varphi(x) h(t)$ substitute

$$\rho_0 \varphi(x) h''(t) = T_0 \varphi''(x) h(t) + \alpha \varphi(x) h(t)$$

$$\rho_0 \frac{h''(t)}{h(t)} = T_0 \frac{\varphi''(x)}{\varphi(x)} + \alpha \qquad \text{Let } c^2 = \frac{T_0}{\rho_0}$$

$$\frac{h''(t)}{c^2 h(t)} = \frac{\varphi''(x)}{\varphi(x)} + \frac{\alpha}{T_0} = -\lambda$$

This leads to the ODES

$$h''(t) = -c^2\lambda h(t) \qquad\qquad \varphi''(x) + \varphi(x)\frac{\alpha}{T_0} = -\lambda \varphi(x)$$

$$h(0) = 0 \qquad\qquad \text{first} \qquad \varphi(0) = 0 \qquad \varphi(L) = 0$$

Solve...

$$\varphi''(x) = \left(-\lambda - \frac{\alpha}{T_0}\right)\varphi(x)$$

Case $-\lambda - \frac{\alpha}{T_0} > 0$ (No eigen functions)

Case $-\lambda - \frac{\alpha}{T_0} = 0$ (No eigenfunctions)

Case $-\lambda-\frac{\alpha}{T_0}<0$ $\qquad \varphi(x) = a\cos\sqrt{\lambda+\frac{\alpha}{T_0}}\,x + b\sin\sqrt{\lambda+\frac{\alpha}{T_0}}\,x$

$\varphi(0) = a = 0 \qquad$ so $\quad a=0$

$\varphi(L) = b\sin\sqrt{\lambda+\frac{\alpha}{T_0}}\,L = 0$

Thus $\sqrt{\lambda+\frac{\alpha}{T_0}}\,L = n\pi$ for $n=1,2,3,\ldots$

Eigenfunctions are $\varphi_n(x) = \sin\frac{n\pi x}{L}$

$\lambda+\frac{\alpha}{T_0} = \left(\frac{n\pi}{L}\right)^2 \qquad \lambda = \left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{T_0}$ $\quad$ since $\alpha<0$ then

positive $\qquad$ positive

so $\lambda>0$

Now solve the other ODE.

$h''(t) = -c^2\lambda h(t)$

$h(0)=0$

$h(t) = a\cos c\sqrt{\lambda}\,t + b\sin c\sqrt{\lambda}\,t$

$h(0) = a = 0 \qquad$ so $a=0$

$h_n(t) = \sin\left(c\sqrt{\left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{T_0}}\;t\right)$

Superposition

$$u(x,t) = \sum_{n=1}^{\infty} a_n \varphi_n(x) h_n(t)$$

$$= \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\,\sin\left(c\sqrt{\left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{T_0}}\;t\right)$$

Solve for the remaining initial condition $\frac{\partial u}{\partial t}(x,0) = f(x).$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} a_n \, c\sqrt{\left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{T_0}} \, \sin\frac{n\pi x}{L} \cos\left(c\sqrt{\left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{T_0}}\, t\right)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} a_n \, c\sqrt{\left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{T_0}} \, \sin\frac{n\pi x}{L} = f(x)$$

By orthogonality integrate over $[0, L]$ after mult by $\sin\frac{m\pi x}{L}$

$$\int_0^L \sum_{n=1}^{\infty} a_n \, c\sqrt{\left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{T_0}} \, \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = \int_0^L f(x) \sin\frac{m\pi x}{L}\, dx$$

only $m=n$

$$\int_0^L a_m c\sqrt{\left(\frac{m\pi}{L}\right)^2 - \frac{\alpha}{T_0}} \, \sin^2\frac{m\pi x}{L}\, dx = \int_0^L f(x) \sin\frac{m\pi x}{L}\, dx$$

$$\frac{L}{2} a_m c\sqrt{\left(\frac{m\pi}{L}\right)^2 - \frac{\alpha}{T_0}} = \int_0^L f(x) \sin\frac{m\pi x}{L}\, dx$$

Thus,

$$a_m = \frac{\frac{2}{L}\int_0^L f(x) \sin\frac{m\pi x}{L}\, dx}{c\sqrt{\left(\frac{m\pi}{L}\right)^2 - \frac{\alpha}{T_0}}}$$

and

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \cdot \sin\left(c\sqrt{\left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{T_0}}\, t\right)$$

not really extra credit...

$$\frac{\partial \rho}{\partial t} + 5t\frac{\partial \rho}{\partial x} = 3\rho \qquad \text{with} \qquad \rho(x,0) = x^2.$$

$x = x(t)$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + x'(t)\frac{\partial \rho}{\partial x}$$

$x'(t) = 5t$

$x(t) = \frac{5}{2}t^2 + x_0$

$$\frac{d\rho}{dt} = 3\rho.$$

$\rho(x(t),t) = \rho(x_0,0)\, e^{3t}$

$\rho(x(t),t) = x_0^2 e^{3t}$

Along the characteristics

$$\rho(\underbrace{\tfrac{5}{2}t^2 + x_0}_{x}, t) = x_0^2 e^{3t}$$

$x = \frac{5}{2}t^2 + x_0$ $\qquad$ $x_0 = x - \frac{5}{2}t^2$

Answer

$$\boxed{\rho(x,t) = \left(x - \tfrac{5}{2}t^2\right)^2 e^{3t}}$$