Solution to the heat equation with insulated boundary.

PDE $\quad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad$ for $x \in [0,L]$, $t \geq 0$

Boundary $\quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \qquad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0 \qquad t \geq 0$

Initial cond. $\quad u(x,0) = f(x)$ for $x \in [0,L]$

Separation of variables $u(x,t) = \varphi(x)G(t)$. Plug in to PDE

$$\varphi(x)G'(t) = k\,\varphi''(x)\,G(t)$$

$$\frac{G'(t)}{kG(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda$$

since the left depend only on t

depends only on x

then these are constant. Call the constant $-\lambda$.

Solve the ODEs.

$G'(t) = -k\lambda G(t)$ and $\varphi''(x) = -\lambda\varphi(x)$

boundary conditions $\varphi'(0)=0$ and $\varphi'(L)=0$

Consider φ first: In cases...

Case $\lambda=0$: $\quad \varphi''=0 \quad$ general solution $\varphi(x) = c_1x + c_2$

$\varphi'(x) = c_1 \qquad \varphi'(0)=0$ means $c_1=0$

$\varphi'(L) = c_1 = 0$

Thus $\varphi(x) = c_2$ is a non-zero solution.

Case $\lambda < 0$: $\quad \varphi''(x) = -\lambda \varphi(x)$

$$\varphi''(x) = |\lambda| \varphi(x)$$

plug in $\varphi(x) = e^{rx}$ to obtain

$$r^2 \cancel{e^{rx}} = |\lambda| \cancel{e^{rx}}$$

$$r^2 = |\lambda| \quad \text{or} \quad r = \pm\sqrt{|\lambda|}$$

general solution: $\varphi(x) = c_1 e^{x\sqrt{|\lambda|}} + c_2 e^{-x\sqrt{|\lambda|}}$

use the boundary conditions $\varphi'(0)=0$ and $\varphi'(L)=0$

$$\varphi'(x) = c_1\sqrt{|\lambda|}\, e^{x\sqrt{|\lambda|}} + c_2\left(-\sqrt{|\lambda|}\right) e^{-x\sqrt{|\lambda|}}$$

$$\varphi'(0) = c_1\sqrt{|\lambda|} - c_2\left(\sqrt{|\lambda|}\right) = 0 \qquad \text{so } c_1 = c_2$$

Therefore

$$\varphi(x) = c_1 e^{x\sqrt{|\lambda|}} + c_1 e^{-x\sqrt{|\lambda|}} = 2c_1\left(\frac{e^{x\sqrt{|\lambda|}} + e^{-x\sqrt{|\lambda|}}}{2}\right)$$

$$= 2c_1 \cosh\left(x\sqrt{|\lambda|}\right)$$

Now

$$\varphi'(x) = 2c_1\sqrt{|\lambda|}\sinh\left(x\sqrt{|\lambda|}\right)$$

$$\varphi'(L) = 2c_1\sqrt{|\lambda|}\,\underbrace{\sinh\left(L\sqrt{|\lambda|}\right)}_{\text{since this is not zero then } c_1 = 0}$$

and all we have is the zero solution.

Case $\lambda > 0$ then $\varphi''(x) = -\lambda\varphi(x)$

general solution $\varphi(x) = c_1 \sin\sqrt{\lambda}x + c_2\cos\sqrt{\lambda}x$

$$\varphi'(x) = c_1\sqrt{\lambda}\cos\sqrt{\lambda}x - c_2\sqrt{\lambda}\sin\sqrt{\lambda}x$$

Boundary conditions

$\varphi'(0) = c_1\sqrt{\lambda} = 0 \quad$ so $\quad c_1 = 0$

$\varphi'(L) = -c_2\sqrt{\lambda}\sin\sqrt{\lambda}L = 0$

thus $\sqrt{\lambda}L = \pi n$ for some $n \in \mathbb{Z}$ and $n \neq 0$ since $\lambda \neq 0$

Thus $\lambda = \frac{\pi^2 n^2}{L^2}$

Solutions

$$\varphi_n(x) = c_1 \sin\sqrt{\lambda}x + c_2\cos\sqrt{\lambda}x$$

$$= c_2 \cos\frac{\pi n}{L}x \quad \text{for } n \in \mathbb{N}$$

{ Since cos is even, the exact same solutions are for negative n's so we skip them

Combining the $\lambda = 0$ case with $\lambda > 0$ gives

$$\varphi_n(x) = c_n \cos\frac{\pi n}{L}x \quad \text{for } n = 0, 1, 2, \ldots$$

Now solve the other ODE

$$G'(t) = -k\lambda G(t) \qquad \lambda = \frac{\pi^2 n^2}{L^2}$$

General solution

$$G(t) = c e^{-k\lambda t} \quad \text{then} \quad G_n(t) = c_n e^{-k\frac{\pi^2 n^2}{L^2}t}$$

Solutions to the PDEs

$$u_n(x,t) = \varphi_n(x)G_n(t) = c_n\left(\cos\frac{\pi n}{L}x\right)e^{-k\frac{\pi^2 n^2}{L^2}t}$$

superposition

$$u(x,t) = \sum_{n=0}^{\infty} A_n\left(\cos\frac{\pi n}{L}x\right)e^{-k\frac{\pi^2 n^2}{L^2}t}$$

Solve for the A_n's to satisfy the initial condition.

Initial cond. $u(x,0) = f(x)$ for $x \in [0,L]$

Solve for the A_n's

$$\sum_{n=0}^{\infty} A_n \cos \frac{\pi n}{L} x = f(x)$$

Trigonometry.

Angle addition

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$

$$\frac{d}{da} \sin(a+b) = \frac{d}{da}(\sin a \cos b + \cos a \sin b)$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

subtract $\cos(a-b) = \cos a \cos b + \sin a \sin b$

$$\cos(a+b) - \cos(a-b) = -2 \sin a \sin b$$

$$\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$$

Instead add to obtain

$$\cos a \cos b = \frac{1}{2}(\cos(a-b) + \cos(a+b))$$

Orthogonality of cosines.

$$\int_0^L \cos \frac{n\pi}{L} x \cos \frac{m\pi}{L} x \, dx =$$

$$= \int_0^L \frac{1}{2}\left(\cos \frac{(n-m)\pi}{L} x + \cos \frac{(n+m)\pi}{L} x\right) dx$$

Note that $n, m \geq 0$ so the only time $n-m=0$ is when $n=m$. But if $n=m=0$ then $n+m=0$ too!

$$= \begin{cases} \frac{L}{2} & \text{if } n=m>0 \\ L & \text{if } n=m=0 \\ 0 & \text{otherwise} \end{cases}$$

Multiply both sides by $\cos\frac{m\pi}{L}x$

$$\int_0^L \sum_{n=0}^{\infty} A_n \cos\frac{\pi n}{L}x \cos\frac{m\pi}{L}x \, dx = \int_0^L f(x) \cos\frac{m\pi}{L}x \, dx$$

Finish next time ...