5 Note that $n, m \geq 0$ so the only time $n-m=0$ is when $n=m$. But if $n=m=0$ then $n+m=0$ too!

$$= \begin{cases} \frac{L}{2} & \text{if } n=m>0 \\ L & \text{if } n=m=0 \\ 0 & \text{otherwise for } n \neq m \text{ and } n \geq 0, m \geq 0 \end{cases}$$

---

Multiply both sides by $\cos \frac{m\pi}{L} x$

$$\int_0^L \sum_{n=0}^{\infty} A_n \cos \frac{\pi n}{L} x \cos \frac{m\pi}{L} x \, dx = \int_0^L f(x) \cos \frac{m\pi}{L} x \, dx$$

Case $m=0$:

$$\int_0^L A_0 = L A_0 = \int_0^L f(x) dx, \qquad A_0 = \frac{1}{L} \int_0^L f(x) dx$$

Case $m>0$:

$$\int_0^L A_m \left(\cos \frac{\pi m}{L}\right)^2 dx = \frac{L}{2} A_m = \int_0^L f(x) \cos \frac{m\pi}{L} x \, dx$$

$$A_m = \frac{2}{L} \int_0^L f(x) \cos \frac{m\pi}{L} x \, dx$$

Therefore the solution to

PDE $\qquad \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad$ for $x \in [0, L]$, $t \geq 0$

B.C. $\qquad \left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \qquad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0 \qquad$ for $t \geq 0$

I.C. $\qquad u(x,0) = f(x) \qquad$ for $x \in [0, L]$

Is given by

$$u(x,t) = \sum_{n=0}^{\infty} A_n \left(\cos \frac{n\pi}{L} x\right) e^{-k\frac{n^2\pi^2}{L^2} t}$$

where

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad \text{and} \quad A_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi}{L} x\, dx$$

### 2.4.2 Heat Conduction in a Thin Insulated Circular Ring

We have investigated a heat flow problem whose eigenfunctions are sines and one whose eigenfunctions are cosines. In this subsection, we illustrate a heat flow problem whose eigenfunctions are both sines *and* cosines.

Let us formulate the appropriate initial boundary value problem if a thin wire (with lateral sides insulated) is bent into the shape of a circle, as illustrated in Fig. 2.4.1. For reasons that will not be apparent for a while, we let the wire have length $2L$ (rather than $L$, as for the two previous heat conduction problems). Since the circumference of a

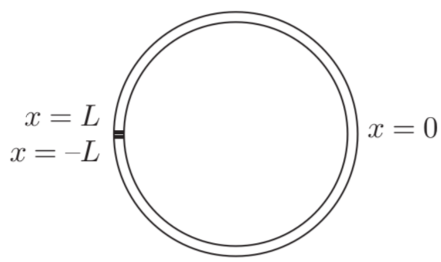
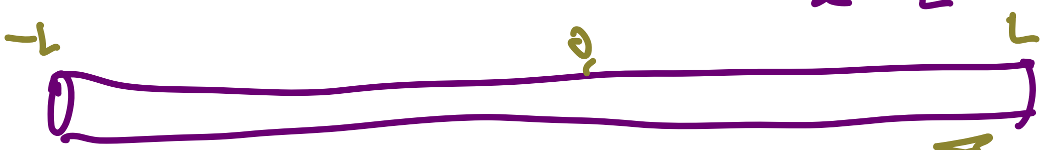



FIGURE 2.4.1 Thin circular ring.

$$u(-L,t) = u(L,t) \qquad \left.\frac{\partial u}{\partial x}\right|_{x=-L} = \left.\frac{\partial u}{\partial x}\right|_{x=L}$$




temperature here is equal $\Longleftrightarrow$ temperature here

Separation of variables.

PDE $\qquad \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad$ for $x \in [-L, L]$, $t \geq 0$

BC. $\qquad u(-L, t) = u(L, t)$

$\qquad \left.\frac{\partial u}{\partial x}\right|_{x=-L} = \left.\frac{\partial u}{\partial x}\right|_{x=L} \qquad$ for $t \geq 0$

IC. $\qquad u(x, 0) = f(x) \qquad$ for $x \in [-L, L]$.

---

Let $\quad u(x,t) = \varphi(x) G(t)$

we get two ODES

$$G'(t) = -k\lambda G(t) \quad \text{and} \quad \varphi''(x) = -\lambda \varphi(x)$$

$$\varphi(-L) = \varphi(L)$$

$$\varphi'(-L) = \varphi'(L)$$

Treat $\varphi$ first. General solution

$$\varphi(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

for $\lambda > 0$

Boundary cond:

$$\varphi(-L) = c_1 \cancel{\cos(\sqrt{\lambda} L)} - c_2 \sin(\sqrt{\lambda} L)$$

$$= \varphi(L) = c_1 \cancel{\cos(\sqrt{\lambda} L)} + c_2 \sin(\sqrt{\lambda} L)$$

Thus $\quad 2 c_2 \sin(\sqrt{\lambda} L) = 0$

either $c_2 = 0$ or $\sqrt{\lambda} L = n\pi$ for $n = 0, 1, \dots$

$$\varphi(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$$

$$\varphi'(x) = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} x) + c_2 \sqrt{\lambda} \cos(\sqrt{\lambda} x)$$

Other boundary condition

$$\varphi'(-L) = c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} L) + c_2 \sqrt{\lambda} \cancel{\cos(\sqrt{\lambda} L)}$$

$$= \varphi(L) = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} L) + c_2 \sqrt{\lambda} \cancel{\cos(\sqrt{\lambda} L)}$$

$$2 c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} L) = 0$$

so either $c_1 = 0$ or $\sqrt{\lambda} L = \pi n$ for $n = 0, 1, \dots$

Therefore

$$(c_2 = 0 \text{ or } \sqrt{\lambda} L = n\pi)$$

$$\text{and } (c_1 = 0 \text{ or } \sqrt{\lambda} L = \pi n)$$

Note if $\sqrt{\lambda} L \neq n\pi$ then $c_2 = 0$ from the first boundary condition and $c_1 = 0$ from the second and that would give the zero solution. Therefore $\sqrt{\lambda}L = n\pi$ or

$$\sqrt{\lambda} = \frac{n\pi}{L} \quad \text{or} \quad \lambda = \frac{n^2\pi^2}{L^2}$$

Thus

$$\phi_n(x) = A_n \cos\frac{n\pi}{L}x + B_n \sin\frac{n\pi}{L}x \quad \text{for } n = 0, 1, \ldots$$

---

The other ODE is the same as before

$$G'(t) = -k\lambda G(t)$$

$$G_n(t) = c_1 e^{-k\lambda t} = c_1 e^{-k\frac{n^2\pi^2}{L^2}t}$$

---

By the superposition principle: Since the PDE is linear and the boundary conditions homogeneous then

$$u(x,t) = \sum_{n=0}^{\infty} c_n \phi_n(x) G_n(t)$$

$$= \sum_{n=0}^{\infty} \left(A_n \cos\frac{n\pi}{L}x + B_n \sin\frac{n\pi}{L}x\right) e^{-k\frac{n^2\pi^2}{L^2}t}$$

combined all the constants into these two.

Thus,

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos\frac{n\pi}{L}x + B_n \sin\frac{n\pi}{L}x\right) e^{-k\frac{n^2\pi^2}{L^2}t}$$

Now, solve for the constants so the initial condition holds true. Thus

$$A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi}{L}x + B_n \sin\frac{n\pi}{L}x\right) = f(x).$$

Lots of orthogonality...

addition

$$\sin(a+b) = \sin a\cos b + \cos a \sin b$$
$$\sin(a-b) = \sin a\cos b - \cos a \sin b$$

$$\frac{d}{da}\sin(a+b) = \frac{d}{da}(\sin a\cos b + \cos a\sin b)$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$

tract $$\cos(a-b) = \cos a\cos b + \sin a\sin b$$

$$(a+b) - \cos(a-b) = -2\sin a\sin b$$

$$\sin a\sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)$$

Finish solving for the constants next time.