Quiz on Wednesday over linear ODEs

Here are some linear ordinary differential equations to practice in preparation for the quiz on Wednesday.

- Solve $xy' + 2y = x^2 - x + 1$ with initial condition $y(1)=1/2$.
- Solve $y' + 3y = 2x - 1$ with initial condition $y(0)=3$.
- Solve $y' + xy = 2x^3$ with initial condition $y(1)=5$.

rate of change of heat energy in time	=	heat energy flowing across boundaries per unit time	+	heat energy generated inside per unit time.

flux

$$\frac{\frac{\partial}{\partial t} A\int_x^{x+\Delta x} e(x,t)\,dx}{\Delta x} = \frac{A\left(\phi(x)-\phi(x+\Delta x)\right)}{\Delta x} + \frac{A\int_x^{x+\Delta x} Q(x,t)\,dx}{\Delta x}$$

Average over x to $x+\Delta x$

$$\lim_{\Delta x\to 0}\frac{A\int_x^{x+\Delta x} Q(x,t)\,dx}{\Delta x} = A\,Q(x,t)$$

$$\lim_{\Delta x\to 0}\frac{A\left(\phi(x)-\phi(x+\Delta x)\right)}{\Delta x} = -A\lim_{\Delta x\to \infty}\frac{\phi(x+\Delta x,t)-\phi(x,t)}{\Delta x} = -A\frac{\partial}{\partial x}\phi(x,t)$$

$$\lim_{\Delta x\to \infty}\frac{\frac{\partial}{\partial t} A\int_x^{x+\Delta x} e(x,t)\,dx}{\Delta x} = \frac{\partial}{\partial t}A\lim_{\Delta x\to \infty}\frac{\int_x^{x+\Delta x} e(x,t)\,dx}{\Delta x} = \frac{\partial}{\partial t}A\,e(x,t)$$

Therefore

$$\frac{\partial}{\partial t}\cancel{A}\,e(x,t) = -\cancel{A}\frac{\partial}{\partial x}\phi(x,t) + \cancel{A}\,Q(x,t)$$

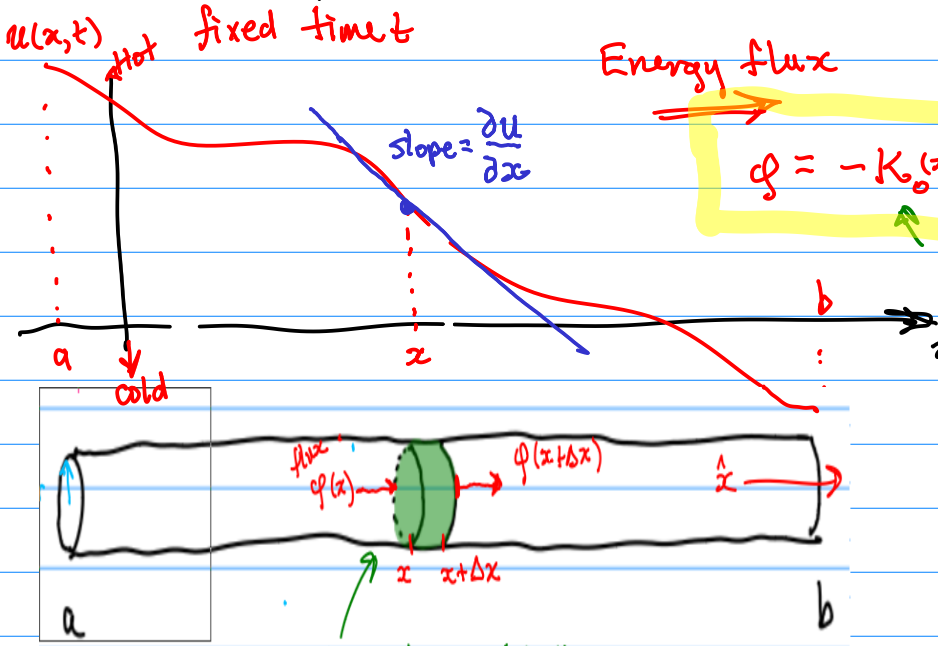
$$\frac{\partial}{\partial t} e(x,t) = -\frac{\partial}{\partial x}\phi(x,t) + Q(x,t)$$

Heat equation in terms of energy... need in terms of temperature...

recall

$$\boxed{e(x,t)} = c(x)\,\rho(x)\,u(x,t) \qquad \frac{\partial}{\partial t} e(x,t) = c(x)\,\rho(x)\,\frac{\partial}{\partial t} u(x,t)$$

How does the flux depend on temperature...



Energy flux

Fourier's law

$$\boxed{\phi = -K_0(x)\,\frac{\partial u}{\partial x}}$$

physical constant "conductivity" of the material..

Fourier's Law: Energy flows from hot to cold in proportion to the difference in temperature...

Fourier's law. Usually (1.2.7) is regarded as one equation in two unknowns, the temperature $u(x,t)$ and the heat flux (flow per unit surface area per unit time) $\phi(x,t)$. How and why does heat energy flow? In other words, we need an expression for the dependence of the flow of heat energy on the temperature field. First we summarize certain qualitative properties of heat flow with which we are all familiar:

1. If the temperature is constant in a region, no heat energy flows.
2. If there are temperature differences, the heat energy flows from the hotter region to the colder region.
3. The greater the temperature differences (for the same material), the greater is the flow of heat energy.
4. The flow of heat energy will vary for different materials, even with the same temperature differences.

$$\frac{\partial}{\partial t} e(x,t) \simeq -\frac{\partial}{\partial x}\varphi(x,t) + Q(x,t)$$

$$\frac{\partial}{\partial t} e(x,t) = c(x)\rho(x)\frac{\partial}{\partial t}u(x,t) \qquad \varphi = -K_0(x)\frac{\partial u}{\partial x}$$

Thus

$$c(x)\rho(x)\frac{\partial}{\partial t}u(x,t) = -\frac{\partial}{\partial x}\left(-K_0(x)\frac{\partial u}{\partial x}\right) + Q(x,t)$$

or

$$c(x)\rho(x)\frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right) + Q(x,t)$$

Heat equation in terms of temperature u

Let's simplify for the first mathematical analysis..

let's suppose the bar is homogeneous... thus

$$c(x)=c, \quad \rho(x)=\rho \quad \text{and} \quad K_0(x)=K_0$$

also suppose $Q=0$.

now K_0 is constant

$$c\rho\frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + \cancel{Q(x,t)}$$

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} \qquad \boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}} \quad \text{where } k = \frac{K_0}{c\rho}.$$

Simplest heat equation...

$$\int v\frac{\partial^2 u}{\partial x^2}\,dx = \text{integrate by parts twice} = \text{stuff} + \int\left(\frac{\partial^2 v}{\partial x^2}\right)u\,dx$$

In terms of linear algebra.

$$x\cdot Ay = Ay\cdot x = (Ay)^T x = y^T A^T x = y\cdot A^T x = A^T x\cdot y.$$

in this analogy it's like $A^T = A$.

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- Solve y' + xy = $2x^3$ with initial condition y(1)=5.

Start solving.

xy' + 2y = x^2−x+1

$$y' + \frac{2}{x}y = x - 1 + \frac{1}{x}$$

$$\boxed{y' + p(x)y = f(x)} \qquad p(x) = \frac{2}{x} \quad \text{and} \quad f(x) = x - 1 + \frac{1}{x}$$

$$y'\mu + xy\mu = \frac{d}{dx}(y\mu)$$

Solve $\underbrace{y' + xy}$ = $2x^3$ with initial condition y(1)=5.

$$\mu(x) = e^{\int_{x_0}^{x} p(s)\,ds} = e^{\int_1^x s\,ds} = e^{\left(\frac{1}{2}s^2\Big|_1^x\right)} = e^{\frac{1}{2}x^2 - \frac{1}{2}}$$

$$\frac{d}{dx}\left[ye^{\frac{1}{2}(x^2-1)}\right] = 2x^3e^{\frac{1}{2}(x^2-1)} = 2e^{-1/2}x^3e^{\frac{1}{2}x^2}$$

integrate both sides $\int_1^x \ldots$

$$y(x)e^{\frac{1}{2}(x^2-1)} - \underbrace{y(1)}_{5}\underbrace{e^{\frac{1}{2}(1^2-1)}}_{1} = \int_1^x 2e^{-1/2}s^3e^{\frac{1}{2}s^2}\,ds$$

$$y(x) = 5e^{-\frac{1}{2}(x^2-1)} + e^{-\frac{1}{2}(x^2-1)}\int_1^x 2e^{-1/2}s^3e^{\frac{1}{2}s^2}\,ds$$

Now work the integral

$$\int_1^x s^3e^{\frac{1}{2}s^2}\,ds = \int_{1/2}^{x^2/2} 2ue^u\,du = (A+Bu)e^u\Big|_{1/2}^{x^2/2}$$

$$u = \frac{1}{2}s^2 \quad du = s\,ds$$

$$\frac{d}{du}(A+Bu)e^u = (A+Bu)e^u + Be^u = (A+B+Bu)e^u = 2ue^u$$

$$A+B=0 \qquad B=2 \qquad A=-2$$

Therefore

$$\int_1^x s^3 e^{\frac{1}{2}s^2}\,ds = (-2+2u)e^u\Big|_{1/2}^{x^2/2} = (-2+x^2)e^{x^2/2} - (-2+1)e^{1/2}$$

And so

$$y(x) = 5\,e^{-\frac{1}{2}(x^2-1)} + e^{-\frac{1}{2}(x^2-1)}\,2e^{-1/2}\left[(-2+x^2)e^{x^2/2} + e^{1/2}\right]$$