homogeneous

PDE: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ $\quad 0 < x < L, \quad t > 0$

homogeneous

BC: $u(0,t) = 0$, $u(L,t) = 0$

Will solve this first

not homogeneous...

IC: $u(x,0) = f(x)$.

page 45 in the text

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

Goal solve for a_n so that I.C. is satisfied

Simple example: What if f were already expressed in term of sine functions:

$$f(x) = 3\sin\left(\frac{\pi x}{L}\right) + 12\sin\left(\frac{5\pi x}{L}\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = 3\sin\left(\frac{\pi x}{L}\right) + 12\sin\left(\frac{5\pi x}{L}\right)$$

$n=1 \qquad a_1 = 3 \qquad\qquad a_n = 0$ for all other values of n.

$n=5 \qquad a_5 = 12$

Solution

$$u(x,t) = \sum_{\boxed{n=1}}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

$n>0$

$$= 3\sin\left(\frac{\pi x}{L}\right) e^{-\left(\frac{\pi}{L}\right)^2 kt} + 12\sin\left(\frac{5\pi x}{L}\right) e^{-\left(\frac{5\pi}{L}\right)^2 kt}$$

How to solve for a_n's given a general initial condition?

use orthogonality...

Orthogonality of $\sin\left(\frac{n\pi x}{L}\right)$ on the interval $[0,L]$ with respect to the dot product (note $n=1,2,3,\ldots$ here)

$$(f,g) = f(x)\cdot g(x) = \int_0^L f(x)g(x)\,dx$$

$$\left(x \to \sin\frac{n\pi x}{L},\ x \to \sin\frac{m\pi x}{L}\right) = \sin\frac{n\pi x}{L} \cdot \sin\frac{m\pi x}{L} = \int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx$$

sin is odd
cos is even

$$\sin(a+b) = \sin a\cos b + \cos a\sin b$$
$$\sin(a-b) = \sin a\cos b - \cos a\sin b$$

$$\frac{\partial}{\partial a}\sin(a+b) = \frac{\partial}{\partial a}\left(\sin a\cos b + \cos a\sin b\right)$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$
$$\cos(a-b) = \cos a\cos b + \sin a\sin b$$

$$\cos(a+b) - \cos(a-b) = -2\sin a\sin b$$

$$\sin a\sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)$$

$$\int_0^L \sin\underbrace{\frac{n\pi x}{L}}_{a}\sin\underbrace{\frac{m\pi x}{L}}_{b}\,dx = \int_0^L \frac{1}{2}\left(\cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}\right)dx$$

Case $m \neq n$. $n-m \neq 0$ also since $m,n > 0$ then $n+m \neq 0$

$$" = \int_0^L \frac{1}{2}\left(\cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}\right)dx$$

$$= \frac{1}{2}\left(\frac{L}{(n-m)\pi}\sin\frac{(n-m)\pi x}{L} - \frac{L}{(n+m)\pi}\sin\frac{(n+m)\pi x}{L}\right)\Bigg|_0^L = 0-0-(0-0) = 0$$

Case $m=n$. Normalization.

$$\int_0^L \frac{1}{2}\left(\cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}\right)dx = \int_0^L \frac{1}{2}\left(\cos\frac{0\,\pi x}{L} - \cos\frac{(2m)\pi x}{L}\right)dx$$

$$= \int_0^L \frac{1}{2}\,dx - \frac{1}{2}\int_0^L \cos\frac{2m\pi x}{L}\,dx = \frac{L}{2} - 0+0 = \frac{L}{2}$$

Use the orthogonality to solve for the a_n's.

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$ initial distribution of heat

$$\int_0^L \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx$$

switch order... needs justification next chapter...

$$\sum_{n=1}^{\infty} a_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx$$

only $m=n$ survives.

$$a_m \frac{L}{2} = \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx$$

Therefore:

$$a_m = \frac{2}{L}\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx$$

and I've solved for the constants...

We will check that $\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$ for this choice of a_n's in some concrete cases and then write $u(x,t)$.