The solution of the PDE

homogeneous

PDE: $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$ $\quad 0 < x < L$, $\; t > 0$

homogeneous

BC: $u(0,t) = 0$, $\; u(L,t) = 0$

We'll solve this first

not homogeneous

IC: $u(x,0) = f(x)$.

the text

Is given by

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}}$$

where $\quad a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$

Example: PDE $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$ for $0 < x < L$, $t > 0$

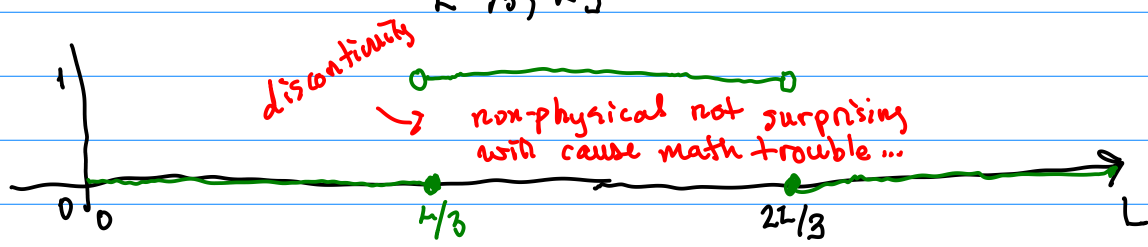
BC $u(0,t) = 0 \cdot u(L,t) = 0$ $\quad t > 0$

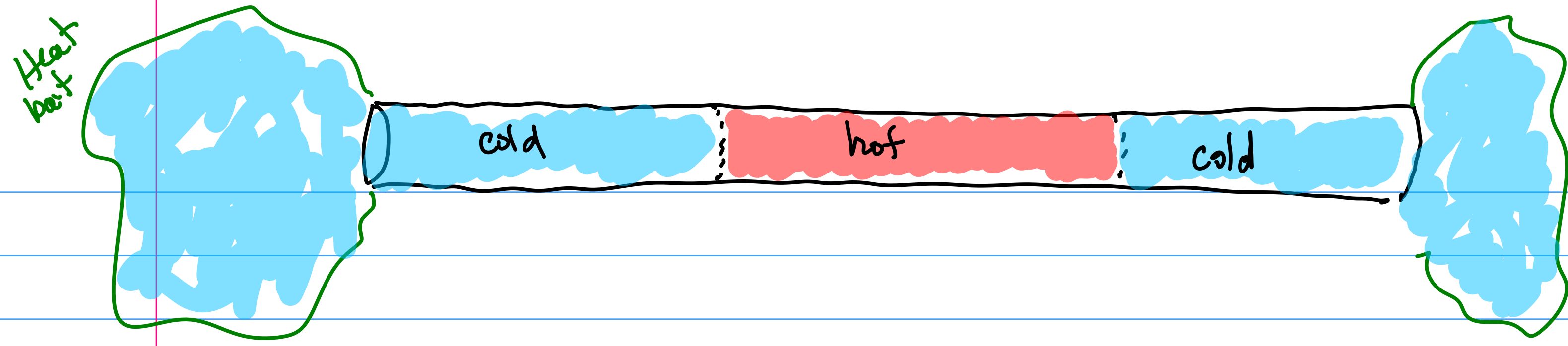
IC $u(x,0) = f(x)$ $\quad 0 < x < L$

Look for a solution that satisfies all of the above and is continuous on $0 \le x \le L$ and $t \ge 0$. Even then we may allow some strange things

$$f(x) = \begin{cases} 0 & \text{for } x \in [0, L/3] \\ 1 & \text{for } x \in (L/3, 2L/3) \\ 0 & \text{for } x \in [2L/3, L] \end{cases}$$

initial temperature





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$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{L/3}^{2L/3} 1\,\sin\frac{n\pi x}{L}\,dx$$

$$= -\frac{2}{\cancel{L}}\frac{\cancel{L}}{n\pi}\cos\frac{n\pi x}{L}\Big|_{L/3}^{2L/3} = \frac{-2}{n\pi}\left(\cos\frac{2n\pi}{3} - \cos\frac{n\pi}{3}\right)$$

$\cos\frac{\pi}{3} = \frac{1}{2}$, $\cos\frac{2\pi}{3} = -\frac{1}{2}$, $\cos\frac{3\pi}{3} = -1$, $\cos\frac{4\pi}{3} = -\frac{1}{2}$, $\cos\frac{5\pi}{3} = \frac{1}{2}$, $\cos\frac{6\pi}{3} = 1$

Could try to simplify and remove all the cosines..

Solution

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

$$\boxed{= \sum_{n=1}^{\infty} \frac{-2}{n\pi}\left(\cos\frac{2n\pi}{3} - \cos\frac{n\pi}{3}\right) \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}}$$

Remark this series is conditionally convergent...

Recall the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Next chapter will discuss how these series are solutions and how they converge...

Next time graph the above solution on computer to visualize it.

Then solve heat eq. with insulating boundary