Insulating boundary ...

The solution of the PDE

homogeneous

PDE: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ $\quad 0 < x < L$, $\ t > 0$

homogeneous

BC: $\frac{\partial}{\partial x}u(0,t) = 0$
$\frac{\partial}{\partial x}u(L,t) = 0$

Will solve this first.

not homogeneous

IC: $u(x,0) = f(x).$

Two ODEs

$G'(t) = -\lambda k G(t)$ and $\varphi''(x) = -\lambda \varphi(x)$

$\varphi'(0) = 0 \quad \varphi'(L) = 0.$

Case $\lambda = 0$: $\quad \varphi_0(x) = C_2$

Case $\lambda < 0$: $\quad$ No - non-zero solution

Case $\lambda > 0$: $\quad \varphi'' = -|\lambda| \varphi$

gen soln: $\quad \varphi(x) = C_1 \cos\sqrt{|\lambda|}\,x + C_2 \sin\sqrt{|\lambda|}\,x$

$\varphi'(x) = -C_1\sqrt{|\lambda|}\sin\sqrt{|\lambda|}\,x + C_2\sqrt{|\lambda|}\cos\sqrt{|\lambda|}\,x$

B.C. $\quad \varphi'(0) = C_2\sqrt{|\lambda|} = 0 \quad$ so $\quad C_2 = 0$

since $\lambda > 0$

$\varphi'(L) = -C_1\sqrt{|\lambda|}\sin\sqrt{|\lambda|}\,L = 0$

$\lambda > 0$

Therefore either $C_1 = 0$ or $\sin\sqrt{|\lambda|}\,L = 0$. If $C_1 = 0$ then $\varphi(x) = 0$ and eigenfunction have to be nonzero.

Thus $\sin\sqrt{|\lambda|}L = 0$ so $\sqrt{|\lambda|}L = \pi n$ for some $n \in \mathbb{Z}$.

since $n=0$ implies $\lambda = 0$, that not right

also if $n<0$ then $L<0$ which is wrong.

so $\sqrt{|\lambda|}L = \pi n$ for $n = 1, 2, 3, \ldots$

$\sqrt{|\lambda|} = \frac{\pi n}{L}$ for $n = 1, 2, 3, \ldots$

a whole sequence $\lambda = \left(\frac{\pi n}{L}\right)^2$

$$\varphi_n(x) = C_1 \cos\frac{\pi n}{L} x \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Now consider the other ODE

$$G'(t) = -\lambda k G(t)$$

gen soln: $G(t) = C_1 e^{-\lambda k t}$

if $\lambda = 0$ then $G_0(t) = C_1$

$\lambda = \left(\frac{\pi n}{L}\right)^2$ then $G_n(t) = C_1 e^{-\left(\frac{\pi n}{L}\right)^2 kt}$

Supperposition is

$$u(x,t) = a_0 \varphi_0(x) G_0(t) + \sum_{n=1}^{\infty} a_n \varphi_n(x) G_n(t)$$

$$= a_0 C_2 C_1 + \sum_{n=1}^{\infty} a_1 C_1' \cos\frac{n\pi}{L} x \; C_1'' e^{-\left(\frac{\pi n}{L}\right)^2 kt}$$

arb const. that also depend on n

Collect the constants together

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi}{L} x \; e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

cosine series ... what we had before was a sine series..

Again use orthonality to solve for the A_n's.

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$

$$\frac{\partial}{\partial a} \sin(a+b) = \frac{\partial}{\partial a}(\sin a \cos b + \cos a \sin b)$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$+ \quad \cos(a-b) = \cos a \cos b + \sin a \sin b$$

$$\cos(a+b) + \cos(a-b) = 2 \cos a \cos b$$

Solve for A_n's to satisfy

IC: $\boxed{u(x,0) = f(x).}$ not homogeneous

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi}{L}x \, e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} = f(x)$$

multiply both sides by $\cos\frac{m\pi x}{L}$ and integrate

$$\cos\frac{m\pi x}{L} A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{m\pi x}{L} \cos\frac{n\pi x}{L} = \cos\frac{m\pi x}{L} f(x)$$

$$\int_0^L \cos\frac{m\pi x}{L} A_0 \, dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos\frac{m\pi x}{L} \cos\frac{n\pi x}{L} \, dx = \int_0^L \cos\frac{m\pi x}{L} f(x) \, dx$$

Let $m=0$

$$\int_0^L A_0\,dx + \sum_{n=1}^{\infty} \underbrace{A_n \int_0^L \cos\frac{n\pi x}{L}\,dx}_{=0} = \int_0^L f(x)\,dx$$

$$L A_0 = \int_0^L f(x)\,dx \quad \text{so} \quad \boxed{A_0 = \frac{1}{L}\int_0^L f(x)\,dx}$$

suppose $m \neq 0$, $m > 0$

$$\underbrace{\int_0^L \cos\frac{m\pi x}{L} A_0\,dx}_{=0} + \sum_{n=1}^{\infty} A_n \int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \int_0^L \cos\frac{m\pi x}{L} f(x)\,dx$$

only $n=m$ term survives in the sum

$$\cos(a+b) + \cos(a-b) = 2\cos a \cos b$$

$$\int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_0^L \left(\cos\frac{(m+n)\pi x}{L} + \cos\frac{(m-n)\pi x}{L}\right) dx$$

$$= \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{2} & \text{if } m = n \end{cases}$$

Therefore

$$A_m \cdot \frac{L}{2} = \int_0^L \cos\frac{m\pi x}{L} f(x)\,dx \quad \text{or} \quad A_m = \frac{2}{L}\int_0^L \cos\frac{m\pi x}{L} f(x)\,dx$$