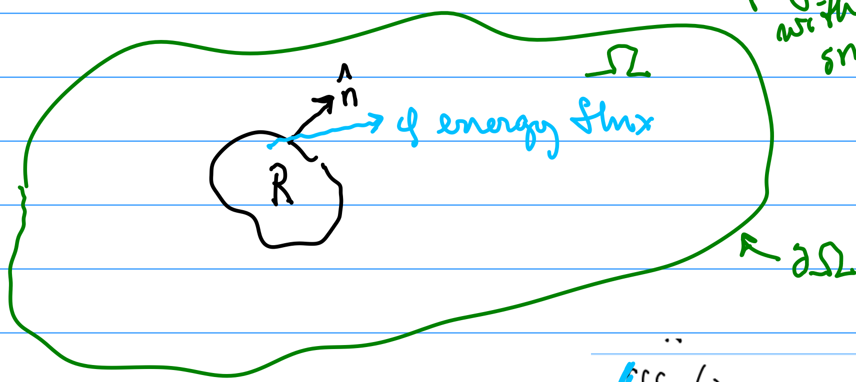
Two dimensional heat equation

regular region with piecewise smooth boundary

Idea, we want to use divergence theorem



$$\iint_R \left(\frac{\partial}{\partial t} e(\vec{x},t) + \nabla\cdot q - Q(\vec{x},t) \right) dS = 0$$

eq. in 2D:

$$c(\vec{x})\rho(\vec{x}) \frac{\partial u(\vec{x},t)}{\partial t} = \nabla\cdot\left(K_0(\vec{x})\nabla u(\vec{x},t)\right) + Q(\vec{x},t)$$

Consider c, ρ and K_0 to constants and $Q \simeq 0$

$$c\rho \frac{\partial u}{\partial t} = K_0 \nabla\cdot\nabla u$$

$$\nabla\cdot\nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$K = \frac{K_0}{c\rho}.$$

$$\boxed{\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}$$

two D heat equation

Consider 2D heat equation.

$$\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{on } (x,y)\in\Omega \text{ and } t>0$$

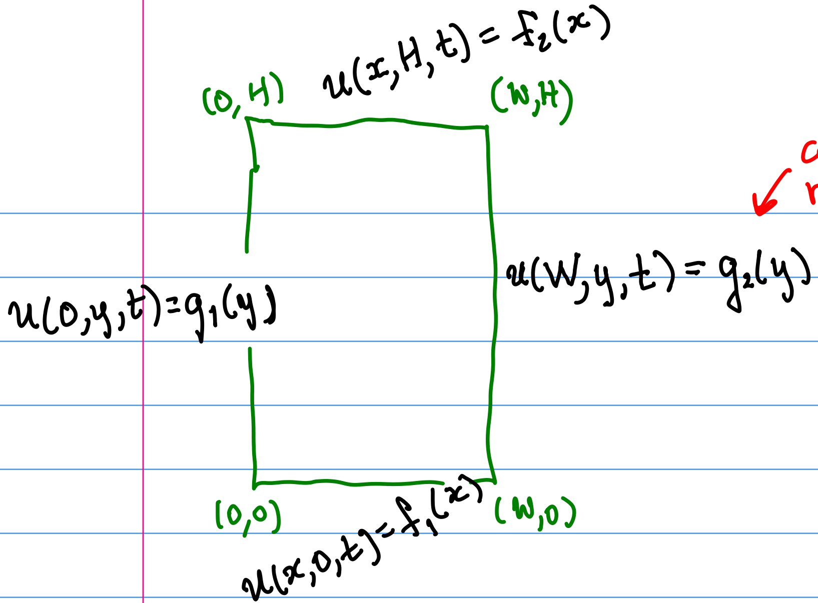
I.C. $u(x,y,0) = u_0(x,y)$ for $(x,y)\in\Omega$.

B.C.

$u(x,y,t) = g(x,y)$ for $(x,y)\in\partial\Omega$ and $t>0$

look for equilibrium solutions. That is $t\to\infty$.

Consider a simple domain (rectangle) Heat bath boundary

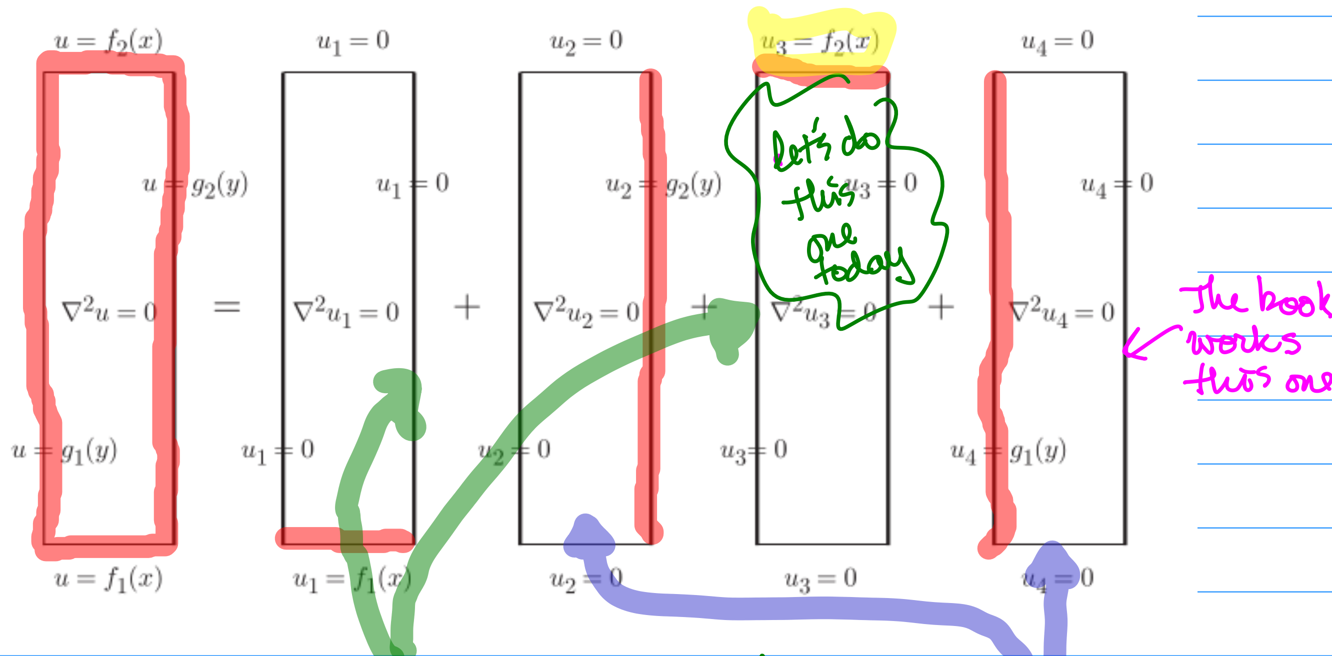


constant in time but not constant along the edge.

equilibrium solution

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$



homogenious along the x direction

homogeneous along the y direction

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \begin{array}{ll} u(x,0) = 0 & u(x,H) = f_2(x) \\ u(0,y) = 0 & u(W,y) = 0 \end{array}$$

Seperate variables

$$u(x,y) = \phi(x)\, G(y)$$

$$0 = \phi''(x) G(y) + \phi(x) G''(y)$$

$$\frac{\phi''(x)}{\phi(x)} = -\frac{G''(y)}{G(y)} = -\lambda$$

no y here — no x here — not depend on either x or y.

ODEs:

$$\frac{\varphi''(x)}{\varphi(x)} = -\lambda$$

$$\varphi''(x) = -\lambda\varphi(x)$$
$$\varphi(0) = 0 \qquad \varphi(W) = 0$$

$$\frac{G''(y)}{G(y)} = \lambda$$

$$G''(y) = \lambda G(y)$$
$$G(0) = 0 \qquad G(H) =$$ use superposition to solve ...

$$\varphi''(x) = -\lambda\varphi(x) \qquad \varphi(0) = 0 \qquad \varphi(W) = 0$$

Gen soln: $\lambda > 0$, $\quad \varphi(x) = c_1 \cos\sqrt{|\lambda|}\, x + c_2 \sin\sqrt{|\lambda|}\, x \qquad \sqrt{\lambda} = \frac{n\pi}{W}$

Satisfy boundary $\quad \varphi(x) = c_2 \sin\frac{n\pi x}{W} \quad$ for $n = 1, 2, \ldots$

$$G''(y) = \lambda G(y) \qquad G(0) = 0$$

Since $\lambda > 0$ Gen soln $\quad G(y) = c_1 e^{\sqrt{\lambda}\, y} + c_2 e^{-\sqrt{\lambda}\, y}$

$$G(0) = c_1 + c_2 = 0 \qquad c_2 = -c_1$$

satisfies one boundary $\quad G(y) = 2c_1 \underbrace{\left(\frac{e^{\sqrt{\lambda}\, y} - e^{-\sqrt{\lambda}\, y}}{2}\right)} = 2c_1 \sinh\sqrt{\lambda}\, y$

Super position

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{W} \sinh\frac{n\pi y}{W}$$

Now solve the inhomogeneous boundary. $u(x,H) = f_2(x)$

$$u(x,H) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{W} \sinh\frac{n\pi H}{W} = f_2(x)$$

solve for the B_n's so the above holds.

Use orthogonality

$$\sum_{n=1}^{\infty} B_n \sin\frac{m\pi x}{W} \sin\frac{n\pi x}{W} \sinh\frac{n\pi H}{W} = \sin\frac{m\pi x}{W} f_2(x)$$

$$\int_0^W \sum_{n=1}^{\infty} B_n \sin\frac{m\pi x}{W} \sin\frac{n\pi x}{W} \sinh\frac{n\pi H}{W}\, dx = \int_0^W \sin\frac{m\pi x}{W} f_2(x)\, dx$$

$$\sum_{n=1}^{\infty} B_n \sinh\frac{n\pi H}{W} \underbrace{\int_0^W \sin\frac{m\pi x}{W} \sin\frac{n\pi x}{W}\, dx}_{= \begin{cases} 0 & \text{if } m \neq n \\ \frac{W}{2} & \text{if } m = n \end{cases}} = \int_0^W \sin\frac{m\pi x}{W} f_2(x)\, dx$$

use this $\sin a \sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)$

$$B_m \sinh\frac{m\pi H}{W} \frac{W}{2} = \int_0^W \sin\frac{m\pi x}{W} f_2(x)\, dx$$

Thus

$$B_m = \frac{2}{W \sinh\frac{m\pi H}{W}} \int_0^W \sin\frac{m\pi x}{W} f_2(x)\, dx$$