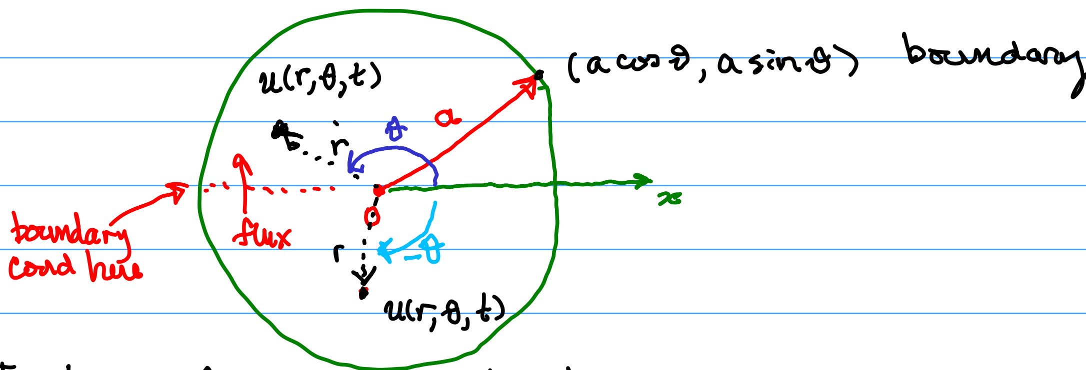
Recall:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Solving the heat equation on the disk. $\frac{\partial u}{\partial t}=k\nabla^2 u$ * u is the temp.



In terms of polar coordinates

PDE $$\frac{\partial u}{\partial t}=k\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$

$\theta\in[-\pi,\pi]$

$r=[0,a]$

$t\geq 0$

IC $$u(r,\theta,0)=u_0(r,\theta)$$

BC Heat bath on boundary: $u(a,\theta,t)=f(\theta,t)$

$$u(r,-\pi,t)=u(r,\pi,t)$$

$$\frac{\partial u}{\partial \theta}(r,-\pi,t)=\frac{\partial u}{\partial \theta}(r,\pi,t)$$

at $r=0$ is where the θ dependency is gone we will assume u is cont. at $r=0$.

Look for equilibrium solutions... No time... boundary conditions don't depend on time... then as $t\to\infty$ the time dependent solution converges to the equilibrium solution.

Laplace eq. on the disc is equilibrium solution

PDE
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \theta \in [-\pi, \pi], \quad r = [0, a]$$

B.C.
$$u(a, \theta) = f(\theta)$$ } not homogeneous.

$$u(r, -\pi) = u(r, \pi)$$
$$\frac{\partial u}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi)$$
} homogeneous (in θ direction) (periodic boundary cond.)

Use separation of variables and superposition to solve.

$$u(r, \theta) = \phi(\theta)\, G(r)$$

$\phi(\theta)$ ← orthogonal eigenfunctions.

Plug it in

$$\frac{\partial^2}{\partial r^2}\Big(\phi(\theta)\, G(r)\Big) + \frac{1}{r}\frac{\partial}{\partial r}\Big(\phi(\theta)\, G(r)\Big) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\Big(\phi(\theta)\, G(r)\Big) = 0$$

$$\phi(\theta)\frac{\partial^2}{\partial r^2}\Big(G(r)\Big) + \phi(\theta)\frac{1}{r}\frac{\partial}{\partial r}\Big(G(r)\Big) + G(r)\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\Big(\phi(\theta)\Big) = 0$$

$$G(r)\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\Big(\phi(\theta)\Big) = -\phi(\theta)\left\{\frac{\partial^2}{\partial r^2}\Big(G(r)\Big) + \frac{1}{r}\frac{\partial}{\partial r}\Big(G(r)\Big)\right\}$$

$$\underbrace{\frac{1}{\phi(\theta)}\frac{\partial^2}{\partial \theta^2}\Big(\phi(\theta)\Big)}_{\text{doesn't depend on } r} = \underbrace{-\frac{r^2}{G(r)}\left\{\frac{\partial^2}{\partial r^2}\Big(G(r)\Big) + \frac{1}{r}\frac{\partial}{\partial r}\Big(G(r)\Big)\right\}}_{\text{doesn't depend on } \theta} = \underbrace{-\lambda}_{\text{const.}}$$

We obtain two ODEs

$$\frac{\partial^2}{\partial\theta^2}\left(\varphi(\theta)\right) = -\lambda\varphi(\theta)$$

$$r^2\left\{\frac{\partial^2}{\partial r^2}\left(G(r)\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(G(r)\right)\right\} = \lambda G(r)$$

$\varphi(-\pi) = \varphi(\pi)$

$\varphi'(-\pi) = \varphi'(\pi)$

$G(a) =$ by super position

$G(0)$ for continuity need to be bounded

boundar for PDE

$u(a,\theta) = f(\theta)$

$u(r,-\pi) = u(r,\pi)$

$\frac{\partial u}{\partial\theta}(r,-\pi) = \frac{\partial u}{\partial\theta}(r,\pi)$

Solve for φ:

Case $\lambda > 0$ $\quad \varphi'' = -\lambda\varphi$

Gen. solution: $\varphi(\theta) = C_1\cos\sqrt{\lambda}\,\theta + C_2\sin\sqrt{\lambda}\,\theta$

$\varphi(-\pi) = C_1\cos\sqrt{\lambda}\,\pi - C_2\sin\sqrt{\lambda}\,\pi$

$\varphi(\pi) = C_1\cos\sqrt{\lambda}\,\pi + C_2\sin\sqrt{\lambda}\,\pi$

If $\varphi(-\pi) = \varphi(\pi)$ then $2C_2\sin\sqrt{\lambda}\,\pi = 0$

$\varphi'(\theta) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}\,\theta + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\,\theta$

$\varphi'(-\pi) = C_1\sqrt{\lambda}\sin\sqrt{\lambda}\,\pi + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\,\pi$

$\varphi'(\pi) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}\,\pi + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\,\pi$

If $\varphi'(-\pi) = \varphi'(\pi)$ then $2C_1\sin\sqrt{\lambda}\,\pi = 0$

Therefore is $\sin\sqrt{\lambda}\,\pi \neq 0$ both $C_1 = 0$ and $C_2 = 0$. Therefore we choose $\sqrt{\lambda} = n$ so that $\sin n\pi = 0$.

Case $\lambda = 0$ $\quad \varphi'' = 0$

gen. soln. $\varphi(\theta) = C_1\theta + C_2$

after boundary cond. we get $C_1 = 0$ so $\varphi(\theta) = C_2$ is another eigen function.

Next solve the other ODE.

$$r^2 \left\{ \frac{\partial^2}{\partial r^2}\left(G(r)\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(G(r)\right) \right\} = \lambda G(r)$$

$$r^2 \frac{d^2}{dr^2}\left(G(r)\right) + r\frac{d}{dr}\left(G(r)\right) - n^2 G(r) = 0 \qquad \text{for } n=0,1,2,\ldots$$

This is an Euler equation. The way to solve... plug in $G = r^m$

$$r^2 \frac{d^2}{dr^2}(r^m) + r\frac{d}{dr}r^m - n^2 r^m = 0$$

$$r^2 m(m-1) r^{m-2} + r m r^{m-1} - n^2 r^m = 0$$

Thus $\quad m(m-1) + m - n^2 = 0$

$$m^2 = n^2 \qquad \text{so} \qquad m = \pm n.$$

General solution $n=1,2,\ldots$

$$G(r) = c_1 r^n + c_2 r^{-n}$$

requiring u continuous at $r=0$ means $c_2 = 0$.

What about $n=0$ (could use reduction of order but easier)

$$\underbrace{r^2 \frac{d^2}{dr^2}\left(G(r)\right) + r\frac{d}{dr}\left(G(r)\right)}_{r\frac{d}{dr}\left(r\frac{d}{dr}G(r)\right)} = 0$$

$$r\frac{d}{dr}\left(r\frac{d}{dr}G(r)\right) = 0$$

Thus $\quad r\frac{d}{dr}G(r) = c_1 \qquad \frac{d}{dr}G(r) = \frac{c_1}{r} \qquad G(r) = c_1 \log r + c_2$