<u>heat equation</u>

$u_t = k u_{xx}$

Example of parabolic PDE
time evolution of a diffusive process.
The heat spreads out

<u>Laplace equation</u>

$u_{xx} + u_{yy} = 0$

Example of elliptic PDE
stationary state of a 2D heat equation. No time.
statics problem

<u>Wave equation.</u>

$u_{tt} = c^2 u_{xx}$

Example of hyperbolic PDE
time evolution of a conservation law.
(no dissipation)

CHAPTER 4

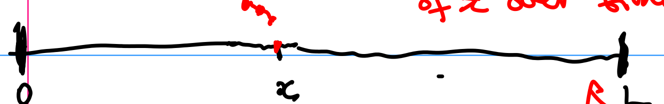
Wave Equation: Vibrating Strings and Membranes

Derivation from Newton's Law $F = ma$

mass
force
acceleration.

Material trajectory of that bit of string marked by x.

$u(x,t)$ tracking the position of x over time.



equilibrium position of a string.

			examples
mass	[M]	grams.	
position	[L]	inches cm	
velocity	[L]/[T]	Mph.	
acceleration	[L]/[T]^2		

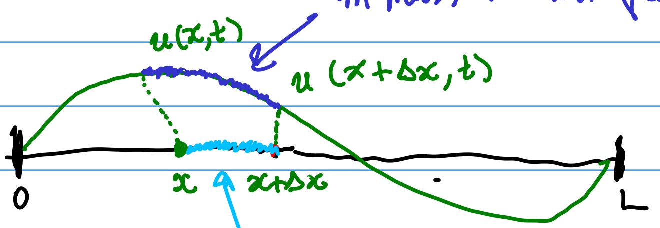
position of the point on the string marked by x at time t is given $u(x,t)$ $\qquad [u] = [L]$

velocity $v = \frac{\partial}{\partial t} u(x,t)$ $\qquad [v] = [L]/[T]$

acceleration $a = \frac{\partial^2}{\partial t^2} u(x,t)$ $\qquad [a] = [L]/[T]^2$

$\rho(x)$ the density of the string at the point x

m mass in that part of the string

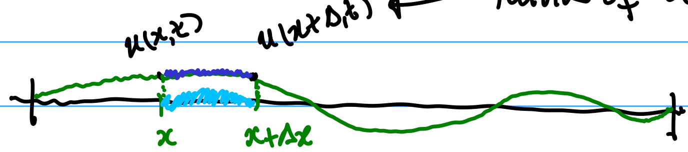


Same as the mass between x and $x+\Delta x$ in the equilibrium position

graph of the position of the string at a fixed time t

$$m = \int_x^{x+\Delta x} \rho(x)\, dx$$

Simplifying assumption the displacement from the equilibrium position is small enough the the material trajectories are only in vertical direction



think of u as a vertical displacement

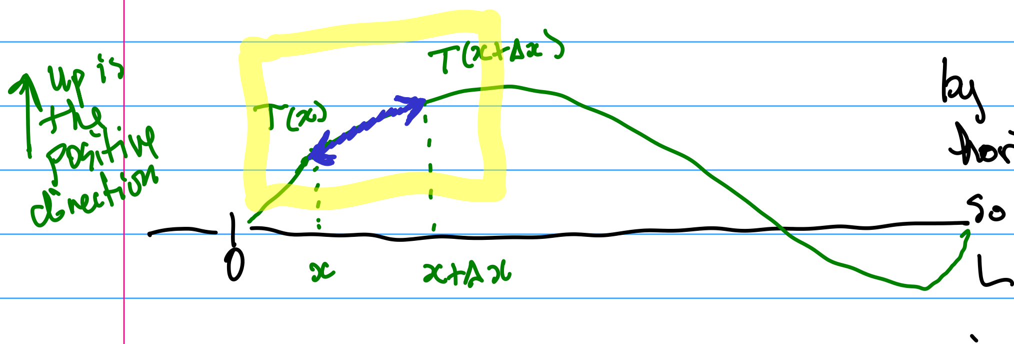
So far

$$F = ma = \left(\int_x^{x+\Delta x} \rho(x)\, dx \right) u_{tt}(x,t) \approx \rho(x)\, \Delta x\, u_{tt}(x,t)$$

we'll take $\Delta x \to 0$ in just a moment and in the limit this is exact.

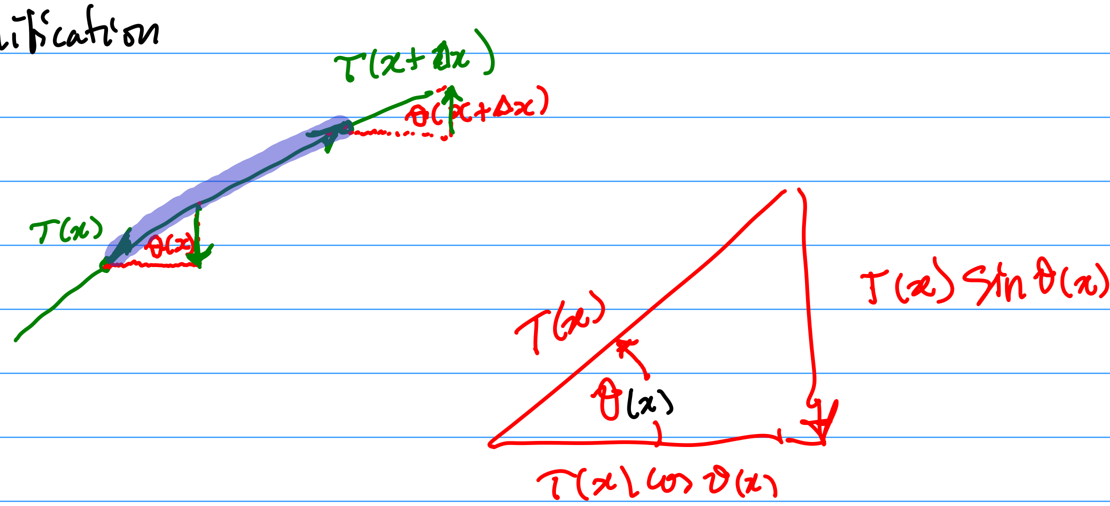
a type of force

Let $T(x)$ be the tension in the string at point x



by assumption there no horizontal movement so so don't worry about the force in the horizontal direction.

Magnification



Vertical component of force on the mass between x and $x+\Delta x$

$$F = T(x+\Delta x)\sin\theta(x+\Delta x) - T(x)\sin\theta(x)$$

$F = ma$ or actually $ma = F$

Therefore

$$\rho(x)\,\Delta x\, u_{tt}(x,t) = T(x+\Delta x)\sin\theta(x+\Delta x) - T(x)\sin\theta(x)$$

Now take the limit as $\Delta x \to 0$

$$\rho(x)\, u_{tt}(x,t) = \lim_{\Delta x \to 0} \frac{T(x+\Delta x)\sin\theta(x+\Delta x) - T(x)\sin\theta(x)}{\Delta x}$$

$$\rho(x)\, u_{tt}(x,t) = \frac{\partial}{\partial x}\left(T(x)\sin\theta(x)\right)$$

recall definition of derivati

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Now take $f(x) = T(x)\sin\theta(x)$
$h = \Delta x$

$$\underbrace{\frac{1}{\Delta x}\left(\int_x^{x+\Delta x} \rho(x)\,dx\right)}_{\substack{\text{average value} \\ \text{of } \rho \text{ on the} \\ \text{interval } x \text{ to } x+\Delta x}} u_{tt}(x,t) = \frac{T(x+\Delta x)\sin\theta(x+\Delta x) - T(x)\sin\theta(x)}{\Delta x}$$

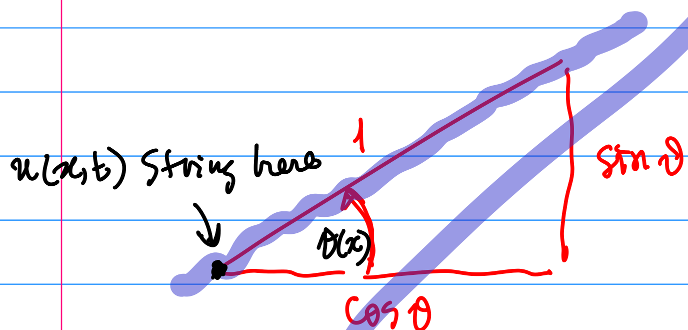
$\Delta x \to 0$

Therefore ...

$$\rho(x)\, u_{tt}(x,t) = \frac{\partial}{\partial x}\left(T(x)\sin\theta(x)\right)$$

write this in terms of u

also might depend on u, but we'll take it constant for simplicity instead...



Slope of the string is $\frac{\partial u(x,t)}{\partial x}$

also given by $\frac{\sin\theta(x)}{\cos\theta(x)}$

$$\frac{\sin\theta(x)}{\cos\theta(x)} = \frac{\partial u(x,t)}{\partial x}$$

Since the displacement is assumed small we might as well assume θ is small

$$\sin\theta(x) = \cos\theta(x)\,\frac{\partial u(x,t)}{\partial x}.$$ then $\cos\theta \approx 1$

$$\sin\theta(x) \approx \frac{\partial u}{\partial x}(x,t).$$

...

$$\rho(x)\, u_{tt}(x,t) = \frac{\partial}{\partial x}\left(T(x)\,\frac{\partial u(x,t)}{\partial x}\right)$$

For the first example we assume $\rho(x) = \rho_0$ a constant and $T(x) = T_0$ is constant.