$$\rho(x)\, u_{tt}(x,t) = \frac{\partial}{\partial x}\left(T(x)\frac{\partial u(x,t)}{\partial x}\right) + \underbrace{\rho(x)\, Q(x,t)}$$

internal forces acting inside the vibrating string...

examples:
- gravity
- electromagnetism
- stiffness of the string
- ⋮

Instead make it simpler.

$Q(x,t)=0$

$T(x) = T_0$ is constant

$\rho(x) = \rho_0$ is constant

Thus..

$$\rho_0\, u_{tt}(x,t) = \frac{\partial}{\partial x}\left(T_0\, u_x(x,t)\right)$$

or

$$u_{tt}(x,t) = \boxed{\frac{T_0}{\rho_0}}\, u_{xx}(x,t)$$

Force divided by a density

$[\rho_0] = [M]/[L]$

$[T_0] = [M]\cdot[a] = [M]\dfrac{[L]}{[T]^2}$ acceleration

$[a] = [u_{tt}] = \dfrac{[L]}{[T]^2}$

$$\left[\frac{T_0}{\rho_0}\right] = \frac{[T_0]}{[\rho_0]} = \frac{\cancel{[M]}\dfrac{[L]}{[T]^2}}{\cancel{[M]}/[L]} = \left(\frac{[L]}{[T]}\right)^2$$

The square of a velocity.

Call that velocity c and set $c^2 = \dfrac{T_0}{\rho_0}$

$$u_{tt}(x,t) = c^2 u_{xx}(x,t)$$

Check consistency:

$[u] = [L]$

$[u_t] = \frac{[L]}{[T]}$ $\qquad$ $[u_x] = \frac{[L]}{[L]} = 1$

$[u_{tt}] = \frac{[L]}{[T]^2}$ $\qquad$ $[u_{xx}] = \frac{1}{[L]}$

$\frac{[L]}{[T]^2} = [c]^2 \frac{1}{[L]}$ implies $[c]^2 = \frac{[L]^2}{[T]^2}$ or $c = \frac{[L]}{[T]}$

a velocity

so consistent.

A simple example. (page 137)

$$u_{tt}(x,t) = c^2 u_{xx}(x,t)$$

displacement

BC1: $\boxed{\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0, \end{aligned}}$ $\qquad$ IC: $\boxed{\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x), \end{aligned}}$

initial velocity.

homogeneous.

- two initial conditions, because to predict the future I need to perform two time integrations because u_{tt} is second degree in time.

Idea: Seperation of variables.

- Use superposition to construct a general solution from a sum of seperable solutions.

Let $u(x,t) = \varphi(x)h(t)$ and plug it in...

$$u_{tt}(x,t) = c^2 u_{xx}(x,t)$$
$$\varphi(x)h''(t) = c^2 \varphi''(x) h(t)$$

$$\underbrace{\frac{h''(t)}{c^2 h(t)}}_{\text{no } x} = \underbrace{\frac{\varphi''(x)}{\varphi(x)}}_{\text{no } t} = -\lambda$$

Obtain two ODEs

$h''(t) = -\lambda c^2 h(t)$ and $\varphi''(x) = -\lambda \varphi(x)$

$\varphi(0) = 0$ and $\varphi(L) = 0$

bndry determined by the superposition

First

$$\varphi''(x) = -\lambda \varphi(x)$$
$$\varphi(0) = 0 \quad \text{and} \quad \varphi(L) = 0$$

$\lambda > 0$

$$\varphi(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$$

$\lambda = 0$
$\lambda < 0$
don't give non-zero solutions.

$\varphi(0) = c_1 = 0 \qquad c_1 = 0$

$\varphi(L) = c_2 \sin\sqrt{\lambda} L = 0 \qquad \sqrt{\lambda} L = n\pi \qquad n = 1, 2, \ldots$

$$\sqrt{\lambda} = \frac{n\pi}{L}.$$

Second

$$h''(t) = -\lambda c^2 h(t)$$

$$h(t) = a\cos c\sqrt{\lambda}\, t + b\sin c\sqrt{\lambda}\, t = a\cos\frac{n\pi c t}{L} + b\sin\frac{n\pi c t}{L}$$

$h_n(t) = a_n \cos\frac{n\pi c t}{L} + b_n \sin\frac{n\pi c t}{L} \qquad \varphi_n(x) = c_n \sin\frac{n\pi x}{L}$

Family of separable solution superposition

$$u(x,t) = \sum_{n=1}^{\infty} \varphi_n(x)\, h_n(t)$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\left(a_n \cos\frac{n\pi ct}{L} + b_n \sin\frac{n\pi ct}{L}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}$$

Solve for A_n's and B_n's using orthogonality and the initial conditions...

IC: $$u(x,0) = f(x)$$ $$\frac{\partial u}{\partial t}(x,0) = g(x),$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi c0}{L} + \cancel{B_n \sin\frac{n\pi x}{L}\sin\frac{n\pi c0}{L}}$$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \qquad A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$u_t(x,t) = \sum_{n=1}^{\infty} A_n\left(\sin\frac{n\pi x}{L}\right)\left(\frac{n\pi c}{L}\right)\left(-\sin\frac{n\pi ct}{L}\right) + B_n\left(\sin\frac{n\pi x}{L}\right)\left(\frac{n\pi c}{L}\right)\left(\cos\frac{n\pi ct}{L}\right)$$

$$u_t(x,0) = \sum_{n=1}^{\infty}\left(\frac{n\pi c}{L}\right) B_n\left(\sin\frac{n\pi x}{L}\right) = g(x)$$

$$\left(\frac{n\pi c}{L}\right) B_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx \qquad B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

For next time fill in details solving for A_n and B_n

and then use the same trigonometry for a physical interpretation of the solution

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \cos\frac{n\pi c t}{L} + B_n \sin\frac{n\pi x}{L} \sin\frac{n\pi c t}{L}$$