4.4.6. For (4.4.1)–(4.4.3), from (4.4.11) show that

$$u(x,t) = R(x - ct) + S(x + ct),$$

where R and S are some functions.

we did this last time

4.4.7. If a vibrating string satisfying (4.4.1)–(4.4.3) is initially at rest, $g(x) = 0$, show that

$$u(x,t) = \frac{1}{2}[F(x - ct) + F(x + ct)],$$

where $F(x)$ is the odd-periodic extension of $f(x)$. [*Hints*:

1. For all x, $F(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$.

2. $\sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$.]

4.4.8. If a vibrating string satisfying (4.4.1)–(4.4.3) is initially unperturbed, $f(x) = 0$, with the initial velocity given, show that

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(\bar{x})\, d\bar{x},$$

where $G(x)$ is the odd-periodic extension of $g(x)$. [*Hints*:

1. For all x, $G(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin \frac{n\pi x}{L}$.

2. $\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$.]

Today **4.4.3.** Consider a slightly damped vibrating string that satisfies

$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ $\quad c^2 = \frac{T_0}{\rho_0}$

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \underbrace{\beta \frac{\partial u}{\partial t}}_{\text{new term. friction}}.$$

$\boxed{\checkmark}$ (a) ~~Briefly~~ explain why $\beta > 0$.

*(b) Determine the solution (by separation of variables) that satisfies the boundar conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

and the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

You can assume that this frictional coefficient β is relatively small ($\beta^2 < 4\pi$ $\rho_0 T_0/L^2$).

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}.$$

$$\rho_0 u_{tt} = T_0 u_{xx} - \beta u_t$$

multiply the equation by u_t

$$\rho_0 u_t u_{tt} = T_0 u_t u_{xx} - \beta (u_t)^2$$

$$\frac{1}{2}\frac{\partial}{\partial t} u_t^2 = \frac{1}{2} 2 u_t \frac{\partial}{\partial t} u_t = u_t u_{tt}$$

$$\rho_0 \frac{1}{2}\frac{\partial}{\partial t} u_t^2 = T_0 u_t u_{xx} - \beta (u_t)^2$$

now integrate the equation over $[0, L]$.

$$\int_0^L \rho_0 \frac{1}{2}\frac{\partial}{\partial t} u_t^2 \, dx = \int_0^L T_0 u_t u_{xx} \, dx - \int_0^L \beta (u_t)^2 \, dx$$

$$\frac{\partial}{\partial t}\left(\underbrace{\frac{\rho_0}{2}\int_0^L u_t^2 \, dx}_{\text{Kinetic energy}}\right) = T_0 \int_0^L u_t u_{xx} \, dx - \beta \int_0^L (u_t)^2 \, dx$$

why u_t is a velocity...

Now ... Integration by parts...

$$\int_0^L u_t u_{xx} \, dx = pq \Big|_0^L - \int_0^L q \, dp$$

$p = u_t \qquad dp = u_{tx} \, dx$

$dq = u_{xx} \, dx \qquad q = u_x$

recall boundary conditions.

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

$$u_t(0,t) = 0 \quad \text{and} \quad u_t(L,t) = 0$$

$$\int_0^L u_t u_{xx}\,dx = u_t(x,t)\,u_x(x,t)\Big|_{x=0}^{L} - \int_0^L \underbrace{u_x u_{tx}}_{\frac{1}{2}\frac{\partial}{\partial t}u_x^2}\,dx$$

$$\frac{\partial (u_x)^2}{\partial t} = 2u_x \frac{\partial u_x}{\partial t} = 2u_x u_{xt}$$

$$\frac{\partial}{\partial t}\left(\frac{\rho_0}{2}\int_0^L u_t^2\,dx\right) = -T_0\int_0^L \frac{1}{2}\frac{\partial}{\partial t}u_x^2\,dx - \beta\int_0^L (u_t)^2\,dx$$

$$\frac{\partial}{\partial t}\left(\underbrace{\frac{\rho_0}{2}\int_0^L u_t^2\,dx}_{\text{kinetic energy}} + \underbrace{\frac{T_0}{2}\int_0^L u_x^2\,dx}_{\text{potential energy.}}\right) = -\beta\int_0^L (u_t)^2\,dx$$

$$\text{Total energy} = E = \frac{\rho_0}{2}\int_0^L u_t^2\,dx + \frac{T_0}{2}\int_0^L u_x^2\,dx$$

$$\frac{\partial E}{\partial t} = -\beta\,\boxed{\int_0^L (u_t)^2\,dx}$$

this is a positive number.

If $\beta > 0$ then $-\beta\int^L (u_t)^2\,dx$ is negative so E is decreasing over time.

Solve $\rho_0 u_{tt} = T_0 u_{xx} - \beta u_t$ where $u(0,t)=0$ and $u(L,t)=0$

and $u(x,0)=f(x)$ and $\frac{\partial u}{\partial t}(x,0)=g(x)$. using separation of vbls.

Plug in $u(x,t) = h(t)\varphi(x)$ to look for separable solutions

$$\rho_0 h'' \varphi = T_0 h \varphi'' - \beta h' \varphi$$

$$\rho_0 h'' \varphi + \beta h' \varphi = T_0 h \varphi''$$

$$\varphi(\rho_0 h'' + \beta h') = T_0 h \varphi''$$

Thus

$$\frac{\rho_0 h'' + \beta h'}{T_0 h} = \frac{\varphi''}{\varphi} = -\lambda$$

Two ODEs

$$\rho_0 h'' + \beta h' = -\lambda T_0 h \qquad \varphi'' = -\lambda \varphi$$

use superposition to determine the initial conditions...

$\varphi(0)=0 \qquad \varphi(L)=0$

Solution

$$\varphi_n(x) = \sin\frac{n\pi x}{L}$$

where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n=1,2,\ldots$

now solve the other ODE

$$\rho_0 h'' + \beta h' = -\lambda_n T_0 h$$

$$\rho_0 h'' + \beta h' + \lambda_n T_0 h = 0$$

linear homogeneous 2nd order ODE
with constant coefficients ...

Guess and check. let $h = e^{rt}$

$h' = re^{rt}$

$h'' = r^2 e^{rt}$

$$\rho_0 r^2 \cancel{e^{rt}} + \beta r \cancel{e^{rt}} + \lambda_n T_0 \cancel{e^{rt}} = 0 \, .$$

$$\underset{a}{\rho_0} r^2 + \underset{b}{\beta} r + \underbrace{\lambda_n T_0}_{c} = 0$$

Solve for r...

$$r = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 \lambda_n T_0}}{2\rho_0}$$

Now make a superposition... next time...