Math 488: Sample Final Version C

This is a closed-book closed-notes no-calculator-allowed in-class exam. Efforts have been made to keep the arithmetic simple. If it turns out to be complicated, that's either because I made a mistake or you did. In either case, do the best you can and check your work where possible. While getting the right answer is nice, this is not an arithmetic test. It's more important to clearly explain what you did and what you know.

1. Indicate in writing that you have understood the requirement to work independently by writing "I have worked independently on this quiz" followed by your signature as the answer to this question.

2. Recall the one-dimensional heat equation with constant thermal properties given by

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} + Q \quad \text{for} \quad t \ge 0 \quad \text{and} \quad x \in [0, L].$$

Here c is the heat capacity, ρ the density, K_0 the conductivity, Q the rate of production of heat energy and u the temperature. Suppose $L = 2$ and $Q/K_0 = 1$. If the initial and boundary conditions satisfy

$$u(x, 0) = \cos(\pi x) \quad \text{for} \quad x \in [0, 2]$$
$$u(0, t) = 3 \quad \text{and} \quad u(2, t) = 1 \quad \text{for} \quad t > 0,$$

find the equilibrium temperature of the rod obtained as $t \to \infty$.

3. Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad t \ge 0 \quad \text{and} \quad x \in [0, L]$$

subject to the homogeneous boundary conditions

$$\frac{\partial u}{\partial x}\bigg|_{x=0} = 0 \qquad \text{and} \qquad u(L, t) = 0.$$

Solve the initial value problem if the temperature is initially

$$u(x, 0) = -2\cos\left(\frac{5\pi x}{2L}\right).$$

4. For the partial differential equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - v_0\frac{\partial u}{\partial x}$$

what ordinary differential equations are implied by the method of separation of variables?

5. Use the method of characteristics to solve

$$\frac{\partial w}{\partial t} + 2\frac{\partial w}{\partial x} = 0 \qquad \text{with} \qquad w(x,0) = \sin x.$$

6. Consider the wave equation

$$\rho_0\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + \alpha u$$

where ρ_0, T_0 and α are constants subject to the homogenous boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0.$$

Solve the initial value problem if $\alpha < 0$ and

$$u(x,0) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = f(x).$$

7. [Extra Credit] Use the method of characteristics to solve

$$\frac{\partial \rho}{\partial t} + 5t\frac{\partial \rho}{\partial x} = 3\rho \qquad \text{with} \qquad \rho(x,0) = x^2.$$